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Conformal symmetries of first-order ordinary differential equations

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Abstract. This paper outlines a technique for determining whether or not a given first-order ordinary differential equation (ODE) is invariant under a one-parameter Lie group of conformal symmetries. The method does not require the form of the Lie group to be specified *a priori*. Instead, the ODE is used to determine the infinitesimal generator of the group. Once it has been ascertained that the ODE has conformal symmetries, the method immediately yields the ODE's general solution.

1. Introduction

It seems paradoxical that all first-order ordinary differential equations (ODEs) of the form

$$y' = \omega(x, y) \tag{1.1}$$

can be solved (i.e., reduced to quadrature) in principle, yet relatively few are solvable in practice. This fact is most easily explained using Lie's theory of symmetries (see any of [1]–[3] for a modern exposition). If

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y \tag{1.2}$$

is the infinitesimal generator of a one-parameter Lie group of point symmetries of (1.1), and if

$$Q(x, y) \equiv \eta(x, y) - \omega(x, y)\xi(x, y) \tag{1.3}$$

is not identically zero, the general solution of (1.1) is

$$\int \frac{dy - \omega(x, y)dx}{Q(x, y)} = c \tag{1.4}$$

where c is an arbitrary constant. The operator X generates point symmetries of (1.1) iff

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y = \xi\omega_x + \eta\omega_y \tag{1.5}$$

which can be re-written as

$$Q_x + \omega Q_y = \omega_y Q. \tag{1.6}$$

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Equation (1.6) has an infinity of non-zero solutions, any one of which would enable (1.1) to be integrated. (In fact, the set of solutions of (1.6) is a finitely-generated module over the algebra of first integrals of (1.1) — see [6].) The characteristic equation for (1.6) is

$$\frac{dx}{1} = \frac{dy}{\omega} = \frac{dQ}{\omega_y Q}. \tag{1.7}$$

Therefore it is usually necessary to solve (1.1) before (1.6) and (1.7) can be solved. However, the reason for using symmetries in the first place is to find the solution of (1.1)!

In the face of the above difficulty, several approaches are possible. The first is to use group classification, i.e., to find families of ODEs that are invariant under the group generated by a particular X. Most elementary methods are based on this idea. A more recent idea, suggested by Olver [1], is to regard (1.1) as an inappropriate reduction of a second-order ODE that has a solvable non-Abelian Lie algebra. The second-order ODE can be solved completely, providing the general solution of (1.1). Here (1.1) is said to have a Type I hidden symmetry [4]. Families of ODEs having a particular hidden symmetry can be classified [4].

Neither of the above methods is helpful when ω is given and (1.1) does not belong to a family of ODEs having known point or hidden symmetries. One option is to try to find a solution of (1.5) or (1.6) using a variety of ansätze for (η, ξ) or Q. This method is fruitful occasionally, but it depends heavily on the skill and persistence of its user.

In the current paper, an elementary method is developed that enables (1.1) to be tested for the presence of a one-parameter Lie group of conformal symmetries. Conformal symmetries are important in many branches of physics; the best-known such symmetries are translations, rotations, homogeneous scalings, and inversions. It is easy to check for the presence of any single type of conformal symmetry by choosing (η, ξ) appropriately. However, the set of conformal symmetries available to ODEs of the form (1.1) is infinite. The method outlined in this paper is useful because it seeks all conformal symmetries simultaneously, rather than restricting attention to particular symmetries. If conformal symmetries are found, (1.1) can be solved immediately.

2. How to find conformal symmetries

The one-parameter Lie group of symmetries generated by an X satisfying (1.5) is conformal (see [1]) iff

$$\xi_x = \eta_y \qquad \xi_y = -\eta_x. \tag{2.1}$$

Then the symmetry condition (1.5) amounts to

$$\eta_x(1+\omega^2) = \xi\omega_x + \eta\omega_y. \tag{2.2}$$

For simplicity, we restrict attention to regions of the (x, y) plane in which all required derivatives of ω exist (i.e., ω is five times continuously differentiable). The method can be used formally even when ω is less smooth than this, although some care is needed. We introduce the complex quantities

$$w = \eta - \mathrm{i}\xi \qquad \bar{w} = \eta + \mathrm{i}\xi \qquad z = x + \mathrm{i}y \qquad \bar{z} = x - \mathrm{i}y \tag{2.3}$$

and the real function

$$\mu(z,\bar{z}) = \tan^{-1}\left\{\omega\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2\mathrm{i}}\right)\right\} \qquad \text{where} \quad \mu \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{2.4}$$

The Cauchy–Riemann conditions (2.1) guarantee that w is an analytic function of z, and that its complex conjugate, $\bar{w}$, is an analytic function of $\bar{z}$. The symmetry condition (2.2) is equivalent to

$$\operatorname{Im}\left\{\frac{\mathrm{i}}{2}\frac{dw}{dz} + w\mu_z\right\} = 0. \tag{2.5}$$

It is convenient to introduce new complex analytic functions

$$\zeta(z) = \int \frac{dz}{w(z)} \qquad \bar{\zeta}(\bar{z}) = \int \frac{d\bar{z}}{\bar{w}(\bar{z})}. \tag{2.6}$$

Note that

$$r \equiv \frac{\zeta(z)+\bar{\zeta}(\bar{z})}{2} = \int \frac{\eta dx - \xi dy}{\xi^2+\eta^2} \tag{2.7}$$

and

$$s \equiv \frac{\zeta(z)-\bar{\zeta}(\bar{z})}{2\mathrm{i}} = \int \frac{\xi dx + \eta dy}{\xi^2+\eta^2} \tag{2.8}$$

are canonical coordinates, because

$$Xr = 0 \qquad Xs = 1. \tag{2.9}$$

The symmetry condition is, in terms of $\zeta(z)$ and $\bar{\zeta}(\bar{z})$,

$$\operatorname{Im}\left\{\frac{\partial}{\partial\zeta}\left(\mu + \frac{\mathrm{i}}{2}\ln\frac{dz}{d\zeta}\right)\right\} = 0. \tag{2.10}$$

After some elementary algebra, the following result is obtained.

The ODE (1.1) admits a one-parameter Lie group of conformal symmetries iff

$$\mu(z,\bar{z}) = F(r) + \tfrac{\mathrm{i}}{2}\ln(\zeta'(z)) - \tfrac{\mathrm{i}}{2}\ln(\bar{\zeta}'(\bar{z})) \tag{2.11}$$

for some real function F and some complex analytic function $\zeta(z)$.

This result could be used as a basis for group classification: choose any analytic function $\zeta(z)$ to obtain X; the most general first-order ODE (1.1) admitting the symmetry group generated by X is

$$y' = \tan\mu(x+iy, x-iy) \tag{2.12}$$

where $\mu(z,\bar{z})$ is given by (2.11) with $F(r)$ an arbitrary real function.

However, as stated in the introduction, our aim is to derive conformal symmetries (where they exist) from a given ω. This can be achieved by repeated differentiation. If the ODE (1.1) admits conformal symmetries then, from (2.11),

$$\mu_{z\bar{z}} = \frac{1}{4}\zeta'(z)\bar{\zeta}'(\bar{z})F''(r) \tag{2.13}$$

and (provided that $\mu_{z\bar{z}} \neq 0$)

$$\frac{(\ln|\mu_{z\bar{z}}|)_{z\bar{z}}}{\mu_{z\bar{z}}} = \frac{(\ln|F''(r)|)''}{F''(r)} \tag{2.14}$$

$$\frac{(\ln|\mu_{z\bar{z}}| + 2\mathrm{i}\mu)_z(\ln|\mu_{z\bar{z}}| - 2\mathrm{i}\mu)_{\bar{z}}}{\mu_{z\bar{z}}} = \frac{\{(\ln|F''(r)|)'\}^2 + 4\{F'(r)\}^2}{F''(r)}. \tag{2.15}$$

Given ω, it is straightforward to calculate $\mu_{z\bar{z}}$ and (if $\mu_{z\bar{z}} \neq 0$)

$$\nu \equiv \frac{(\ln|\mu_{z\bar{z}}|)_{z\bar{z}}}{\mu_{z\bar{z}}} \tag{2.16}$$

$$\lambda \equiv \frac{(\ln|\mu_{z\bar{z}}| + 2\mathrm{i}\mu)_z(\ln|\mu_{z\bar{z}}| - 2\mathrm{i}\mu)_{\bar{z}}}{\mu_{z\bar{z}}}. \tag{2.17}$$

There are three possibilities.

Case I: ν not constant. If (1.1) admits conformal symmetries, then ν is a function of r and so

$$\frac{\nu_z}{\nu_{\bar{z}}} = \frac{\zeta'(z)}{\bar{\zeta}'(\bar{z})}. \tag{2.18}$$

If the left-hand side of (2.18) is separable, $\zeta'(z)$ can be determined up to an arbitrary real constant factor; otherwise (1.1) has no conformal symmetries. The effect of the arbitrary factor is to multiply X by a constant, so the factor can take any convenient (non-zero) value without loss of generality. Then the canonical coordinates r and s can each be determined (up to an irrelevant constant) by quadrature. The necessary condition that (2.18) should be solvable does not guarantee that (1.1) has conformal symmetries; one must check that $\zeta(z)$ also satisfies the sufficient condition (2.11). It may be easier to check the equivalent condition (2.10), which can be re-written as

$$\mathrm{Im}\left\{\frac{1}{\zeta'(z)}\left(\mu_z - \frac{\mathrm{i}\zeta''(z)}{2\zeta'(z)}\right)\right\} = 0. \tag{2.19}$$

Having ascertained that $\zeta(z)$ satisfies the sufficient condition, it is then easy to determine $F(r)$ from (2.11).

Case II: ν constant, $\mu_{z\bar{z}} \neq 0$. If $\nu = c$ (where c is a real constant) and (1.1) admits conformal symmetries, then

$$(\ln|F''(r)|)' = cF'(r) + d \tag{2.20}$$

where d is a real constant. Then, from (2.15) and (2.17),

$$\lambda = \frac{(c^2+4)\{F'(r)\}^2 + 2cdF'(r) + d^2}{F''(r)}. \tag{2.21}$$

It is straightforward to check that λ is not constant for any solution $F(r)$ of (2.20). Therefore if (1.1) admits conformal symmetries

$$\frac{\lambda_z}{\lambda_{\bar{z}}} = \frac{\zeta'(z)}{\bar{\zeta}'(\bar{z})}. \tag{2.22}$$

As in Case I, $\zeta'(z)$ can be determined up to an irrelevant constant factor. Once again, the sufficient condition (2.19) must also be satisfied.

Case III: $\mu_{z\bar{z}} = 0$. Here μ is of the form

$$\mu = f(z) + \bar{f}(\bar{z}) \tag{2.23}$$

for some analytic function $f(z)$. The ODE (1.1) always admits a conformal symmetry in this case, with

$$\zeta(z) = \int \exp\{-2\mathrm{i}f(z)\}dz. \tag{2.24}$$

Then the sufficient condition (2.11) holds, and $F(r) = 0$.

3. The general solution of the ODE

Once $\zeta(z)$ and $F(r)$ have been found by the above method, it is easy to write down the general solution of (1.1), which is

$$\int \frac{dy - \omega dx}{\eta - \omega\xi} = c \tag{3.1}$$

where c is an arbitrary real constant. In terms of $(\zeta, \bar{\zeta})$,

$$\int \frac{dy - \omega dx}{\eta - \omega\xi} = \frac{1}{\mathrm{i}} \int \frac{e^{-\mathrm{i}F(r)}d\zeta - e^{\mathrm{i}F(r)}d\bar{\zeta}}{e^{-\mathrm{i}F(r)} + e^{\mathrm{i}F(r)}}. \tag{3.2}$$

Therefore in canonical coordinates, the general solution of (1.1) is

$$s - \int \tan(F(r))dr = c. \tag{3.3}$$

If $\mu_{z\bar{z}} = 0$, the general solution of (1.1) is $s = c$; from (2.24), we obtain

$$\mathrm{Im}\left\{\int \exp\{-2\mathrm{i}f(z)\}dz\right\} = c. \tag{3.4}$$

The solution (3.4) is listed by Kamke [5], whereas the more general solution (3.3) of Cases I and II is new.

4. Examples

Here the method is illustrated in Cases I and II, where a little work is needed to determine whether or not conformal symmetries exist. Case III is trivial, and is therefore omitted.

Consider the ODE

$$y' = \frac{x - y(x^2 - y^2)^2}{y + x(x^2 - y^2)^2} = \frac{4(z + \bar{z}) + \mathrm{i}(z - \bar{z})(z^2 + \bar{z}^2)^2}{-4\mathrm{i}(z - \bar{z}) + (z + \bar{z})(z^2 + \bar{z}^2)^2}. \tag{4.1}$$

A short calculation gives

$$\mu = \tan^{-1}\left(\frac{4}{(z^2 + \bar{z}^2)^2}\right) + \frac{\mathrm{i}}{2}\ln\left(\frac{z}{\bar{z}}\right). \tag{4.2}$$

By taking repeated derivatives as described in section 2 (using computer algebra), it is found that (4.1) is a Case I ODE, and that (2.18) amounts to

$$\frac{\zeta'}{\bar{\zeta}'} = \frac{z}{\bar{z}}. \tag{4.3}$$

Therefore

$$\zeta = z^2 \tag{4.4}$$

(to within an arbitrary real constant factor). Comparing (4.2) with (2.11), it is immediately obvious that the ODE (4.1) admits conformal symmetries, and that

$$F(r) = \tan^{-1}(r^{-2}). \tag{4.5}$$

Hence the general solution of (4.1) is

$$s + \frac{1}{r} = c \tag{4.6}$$

where

$$s = \frac{(z^2 - \bar{z}^2)}{2\mathrm{i}} = 2xy \qquad r = \frac{(z^2 + \bar{z}^2)}{2} = x^2 - y^2. \tag{4.7}$$

As a second example, consider the ODE

$$y' = \frac{2xy + (x^2 - y^2)\tan\left\{\frac{2x^2}{(x^2+y^2)^2}\right\}}{x^2 - y^2 - 2xy\tan\left\{\frac{2x^2}{(x^2+y^2)^2}\right\}}. \tag{4.8}$$

Here

$$\mu = \frac{2x^2}{(x^2 + y^2)^2} + \tan^{-1}\left\{\frac{2xy}{x^2 - y^2}\right\} = \frac{(z + \bar{z})^2}{2z^2\bar{z}^2} - \mathrm{i}\ln\left(\frac{z}{\bar{z}}\right) \tag{4.9}$$

and so

$$\mu_{z\bar{z}} = \frac{1}{z^2\bar{z}^2} \quad \text{and} \quad \nu = 0. \tag{4.10}$$

Therefore the ODE (4.8) is an example of Case II. From (2.17),

$$\lambda = \frac{4(z+\bar{z})^2}{z^2\bar{z}^2} \tag{4.11}$$

and therefore

$$\frac{\lambda_z}{\lambda_{\bar{z}}} = \frac{\bar{z}^2}{z^2}. \tag{4.12}$$

Equation (4.12) is separable, and we obtain

$$\zeta = \frac{1}{z} \tag{4.13}$$

(to within an arbitrary real constant factor). Note that

$$\frac{1}{\zeta'(z)}\left(\mu_z - \frac{\mathrm{i}\zeta''(z)}{2\zeta'(z)}\right) = \frac{z+\bar{z}}{z\bar{z}} \tag{4.14}$$

and so the sufficient condition (2.19) is satisfied. Comparing (4.9) with (2.11), we find that

$$F(r) = 2r^2. \tag{4.15}$$

Therefore the general solution of (4.8) is

$$s - \int \tan(2r^2)dr = c \tag{4.16}$$

where

$$s(x, y) = \frac{-y}{x^2+y^2} \qquad r(x, y) = \frac{x}{x^2+y^2}. \tag{4.17}$$

Finally, consider the ODE

$$y' = \frac{\cos x - (y - \ln(\cos x))\sin x}{\sin x + (y - \ln(\cos x))\cos x} \qquad x \in (0, \frac{\pi}{2}). \tag{4.18}$$

A short calculation gives

$$\mu = \tan^{-1}\left\{\ln\left(\frac{e^{\mathrm{i}z} + e^{-\mathrm{i}\bar{z}}}{2}\right)\right\} + \frac{\pi}{2} - \frac{z+\bar{z}}{2}. \tag{4.19}$$

Following the method of section 2, it is found that (4.18) is a Case I ODE, which admits conformal symmetries with

$$\zeta(z) = e^{\mathrm{i}z} \qquad F(r) = \tan^{-1}\{\ln r\}. \tag{4.20}$$

In fact, it is fairly easy to spot (4.20) by comparing (4.19) with (2.11), but the method works just as well whether or not solutions can be found by inspection. Carrying out the quadrature in (3.3), and writing (r, s) in terms of (x, y), we obtain the general solution

$$e^{-y}\left[\sin x + [y + 1 - \ln(\cos x)]\right] = c. \tag{4.21}$$

The above examples have been tested using the *ode* collection of solvers which are available within the computer algebra system MACSYMA. The first example, (4.1), has a polynomial integrating factor which *ode* is able to find. However, *ode* was unable to solve (4.8) and (4.18), suggesting that the current method is a useful addition to existing techniques.

5. Conclusions

The method developed above is straightforward to use, and extends the range of techniques for integrating first-order ODEs. There are only a few cases to consider, and these are easy to check with the aid of computer algebra. Indeed, the whole technique could be implemented as a computer algebra package without much difficulty.

In this paper we have restricted attention to conformal symmetries, in view of their importance in physics. It remains to be seen how far the constructive approach can be used to determine other symmetries of a given first-order ODEs.

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